Lecture 26

Review

Let $f = f(x,y)$, and $g = g(x,y,z)$.

- A level curve of f is the graph of the equation $f(x,y) = k$, where k is a constant.

  Likewise, level surface of g is the graph of the equation $g(x,y,z) = k$, where k is a constant.

- The gradient of:
  - f is $\nabla f = \langle f_x, f_y \rangle$
  - g is $\nabla g = \langle g_x, g_y, g_z \rangle$

- Geometric interpretation of the gradient
  - ∇f is perpendicular to level curves of f
  - ∇g is perpendicular to level surfaces of g
  - given a level surface of g, $g = k$, and a point on it, say (a,b,c), $\nabla g(a,b,c)$ is normal to the tangent plane at (a,b,c), and parallel to the normal line at (a,b,c)

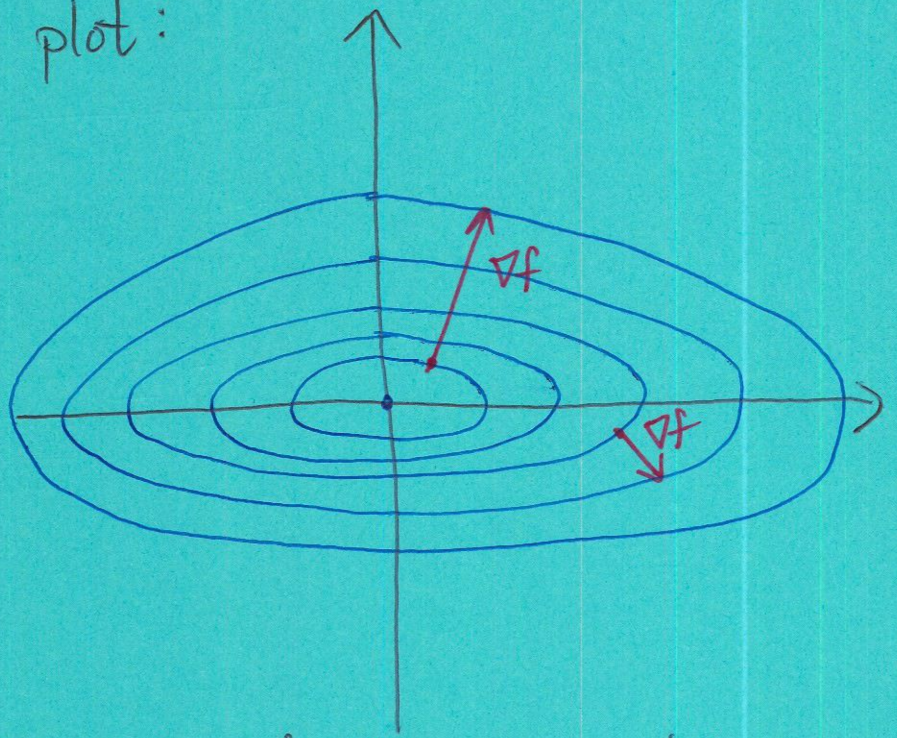
i.e., * The tangent plane to $g(x,y,z)=k$ at (a,b,c) is

$$\boxed{\nabla g(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0}$$

* The normal line to $g(x,y,z)=k$ at (a,b,c) is

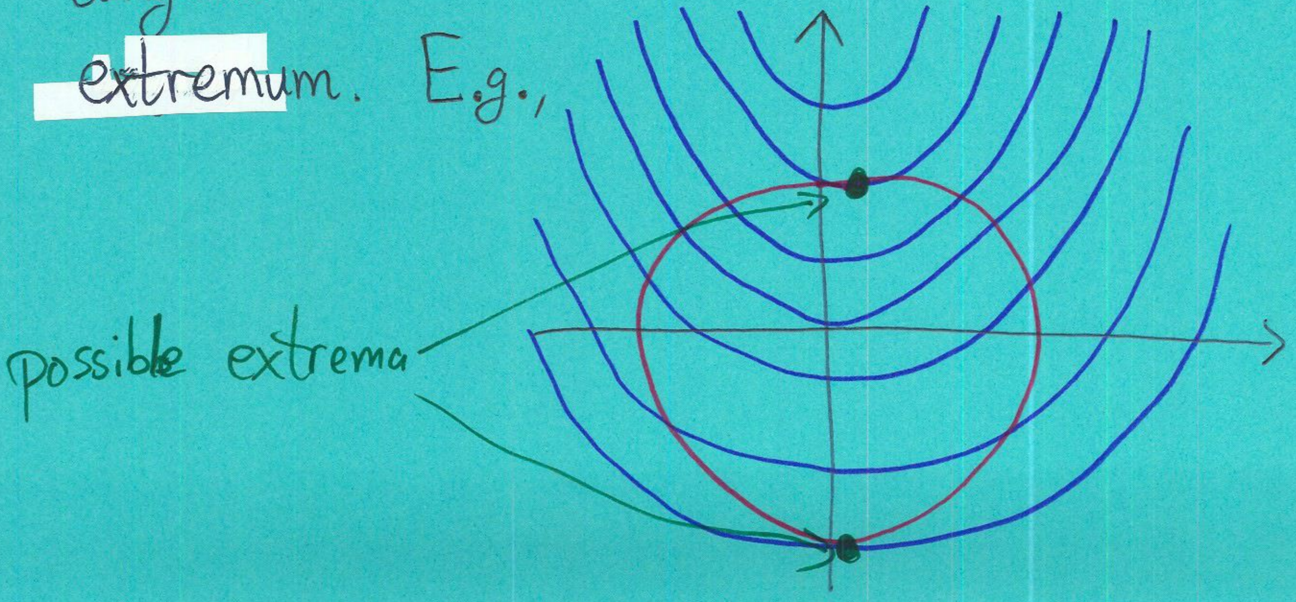
$$\boxed{\vec{l}(t) = \langle a,b,c \rangle + t\,\nabla g(a,b,c)}$$

- Estimating the gradient of f at a point given a contour plot:



∇f is always perpendicular to level curves. The length depends on the density of level curves at the point you're estimating at (more dense = longer).

- Lagrange multipliers
  - To find extrema of f subject to $g=k$, solve the system $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$ (2 or 3 variables)
  - To find extrema of f subject to $g=k$ & $h=c$, solve the system $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = k \\ h = c \end{cases}$ (3 variables only)
  - Visually, in the 2 variable case, the condition $\nabla f = \lambda \nabla g$ means that the curve $g=k$ is tangent to some level curve of f at an extremum. E.g.,



- Directional derivative of f in the direction of $\vec{v}$. First, if $\vec{v}$ is not a unit vector, let $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$, then $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

- The direction of greatest increase of a function f is in the direction of ∇f.

- If $f(x,y)$ is continuous at (a,b), then
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- An easy way to show that a limit does not exist is to check along $y=mx$. This isn't guaranteed to work. It works if the limit depends on m.

- To take partial derivatives, consider all variables other than the one you're taking the derivative with respect to.

- Chain rule: $h(x_1, \cdots, x_n) = f \circ \vec{G}(x_1, \cdots, x_n)$
$f = f(y_1, \cdots, y_m)$, $\vec{G}(x_1, \cdots, x_n) = \langle y_1(x_1, \cdots, x_n), \cdots, y_m(x_1, \cdots, x_n)\rangle$
$$\frac{\partial h}{\partial x_j} = \nabla f(\vec{G}(x_1, \cdots, x_n)) \cdot \frac{\partial \vec{G}}{\partial x_j}(x_1, \cdots, x_n)$$

- You could also use trees to compute by the chain rule
- The chain rule can be used for implicit differentiation, as well as related rates problems
- Second derivative test:
  1) Search for critical points of f, i.e., (a,b) such that $\nabla f(a,b) = \vec{0}$.
  2) Let $D(a,b) = \det \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}$

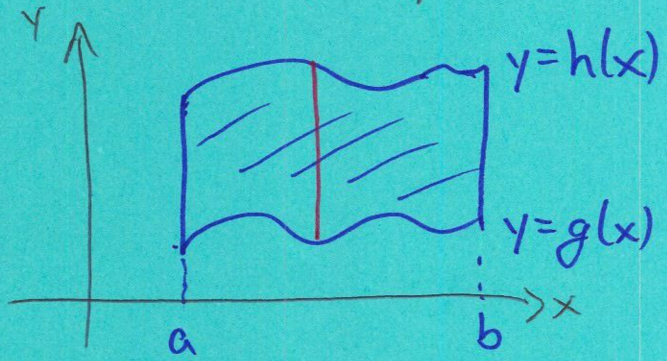
     i) If $f_{xx}(a,b) > 0$ and $D(a,b) > 0$, then (a,b) is a local minimum.

     ii) If $f_{xx}(a,b) < 0$ and $D(a,b) > 0$, then (a,b) is a local maximum.

     iii) If $D(a,b) < 0$, then (a,b) is a saddle point.
- To find extrema on closed and bounded regions, use the second derivative test on critical points inside the region, then search for extreme values on the boundary.

- To compute the integral $\iint_D f(x,y)\,dA$, first graph the region, then decide whether it is easier to integrate with respect to x or y first.
  - Y first is easier if vertical slices work better

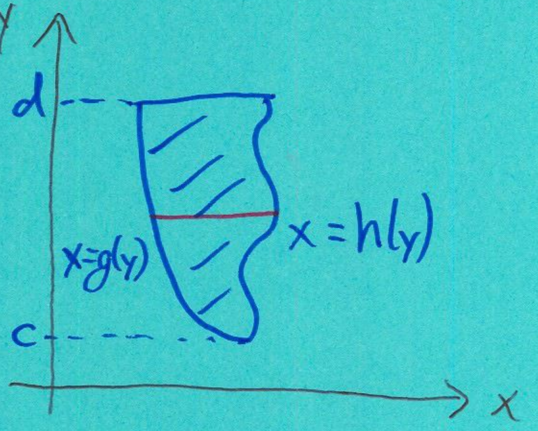
e.g.,



$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

  - X first is easier if horizontal slices work better.

e.g.,



$$\int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

- Be able to read the region of integration from the integral.
  - $\int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy \Rightarrow D = \{(x,y) \mid g(y) \le x \le h(y),\ c \le y \le d\}$
  - $\int_a^b \int_{g(x)}^{h(y)} f(x,y)\,dy\,dx \Rightarrow D = \{(x,y) \mid g(x) \le y \le h(x),\ a \le x \le b\}$